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Curves and proximity on rational surface singularities

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Abstract

We study the germs of curves in a rational surface singularity (S,P) from the point of view of proximity, classifying them up to a notion of equisingularity. We introduce the concept of cluster of infinitely near points and we use it to generalize the Hoskin–Deligne formula, and to give an algorithm to describe a minimal system of generators of a complete ideal in the local ring $\mathcal{O}_{S,P}$. © 1997 Elsevier Science B.V.

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0. Introduction

In order to classify the irreducible plane curve singularities, several invariants have been introduced such as characteristic pairs, multiplicity sequence, value semigroup, etc. In fact, a geometric approach, based on the idea of proximity, was already developed by Enriques in [8] (1915). Recently, this notion of proximity has been applied in [4, 15, 3]. In this paper, we study the germs of curves embedded in a rational surface singularity from the point of view of proximity.

We classify the germs of reduced curves in a rational surface singularity (S,P) up to a notion of equisingularity which generalizes the equisingularity of germs of plane curves. The equisingularity class of such a germ of curve C in (S,P) consists of the weighted dual graph of the minimal embedded desingularization of C in (S,P), together with some weighted arrows corresponding to the branches of C. We express this combinatorial object in terms of some invariants of the singularity (S,P) and the

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curve C, namely, the proximity matrix (Definition 1.4), the intersection matrix in terms of total transforms (1.12) and the orders of C (Definition 1.6). After discussing these invariants in Section 1, we prove that the equisingularity class of C in (S,P) determines the equiresolution class of C (Theorem 2.10). We give an example to show that the converse is not true.

The idea of studing families of Cartier and Weil divisors on (S,P) going through a finite set of points infinitely near P with assigned orders is developed in Section 3. We introduce the notion of cluster with origin at P and generalize the geometric theory of Enriques to rational surface singularities. When we deal with families of Cartier divisors, this allows us to identify the $\mathfrak{m}$-primary complete ideals of the local ring $\mathcal{O}_{S,P}$ with some specific clusters: the Cartier clusters. Using this characterization, we generalize to rational surface singularities the formula given by Hoskin and Deligne [10,6]. This formula computes the minimal number of generators $\mu(I)$ of any $\mathfrak{m}$-primary complete ideal I. In particular, we observe that, as it happens in the nonsingular case, $\mu(I)$ only depends on the orders at the origin of the cluster associated to I. Finally, as another application of the notion of cluster, we give an algorithm to describe a minimal system of generators of I, generalizing to rational surface singularities the procedure given by Casas [5].

1. Constellations of points infinitely near the point P of the rational surface singularity (S,P)

In this section, after recalling the basic properties of rational surface singularities that will be used further, we introduce some definitions and notations and prove some preliminary results. Throughout this paper, a *surface singularity* is a pair (S,P) consisting of the spectrum $S = Spec\,R$ of a noetherian normal complete two-dimensional local ring R containing an algebraically closed field $\mathbf{k}$ isomorphic to its residue field, and the closed point P of S.

1.1. Recall that a surface singularity (S,P) is said to be a *rational surface singularity* if there exists a desingularization $p : X \to S$ such that the stalk at P of $R^1p_*\mathcal{O}_X$ is zero. Moreover, one can prove that any desingularization $p : X \to S$ of (S,P) is a product of blowing ups centered at closed points, and the stalk at P of $R^1p_*\mathcal{O}_X$ is zero. In particular, if P is nonsingular, then (S,P) is a rational surface singularity.

The following properties hold for a rational surface singularity (S,P):

(a) For any Weil divisor C on (S,P) there exists an integer r such that rC is a Cartier divisor on (S,P).

(b) Let $p : X \to S$ be a desingularization of (S,P) and let $\{E_i\}_{i=1}^n$ be the irreducible components of the exceptional locus of p. If D is a divisor on X with $D.E_i = 0$ for all i, then there exists an element h in the maximal ideal of $\mathcal{O}_{S,P}$ such that $(h)^* = D$, where $(h)^*$ is the total transform on X of the divisor given by h on (S,P).

A proof of those results may be found in [1, 2, 14].

1.2. Definition. Let (S,P) be a rational surface singularity. The closed points in the exceptional locus of the blowing up π_1 of P are called *points in the first infinitesimal neighbourhood of* P. For $i > 1$, we define inductively the *points in the ith infinitesimal neighbourhood of* P to be the closed points in the $(i-1)$th infinitesimal neighbourhood of some point in the first infinitesimal neighbourhood of P. The points in some infinitesimal neighbourhood of P are called *points infinitely near* P (see [8]).

A *constellation* $\mathscr{C}$ *of points infinitely near* P (or constellation with origin at P) is a finite set of points infinitely near P containing P and every point preceding a point in $\mathscr{C}$, i.e. if $Q \in \mathscr{C}$ and Q is infinitely near a closed point R, then $R \in \mathscr{C}$.

1.3. We may label the points in $\mathscr{C}$, say $\mathscr{C} = \{P_1, \ldots, P_m\}$, in such a way that $P_1 = P$ and if P_j is infinitely near P_i then $j > i$. In this way, we get a sequence of point blowing ups

$$S_{\mathscr{C}} = S_m \xrightarrow{\pi_m} S_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S, \tag{1}$$

where π_i is the blowing up with center P_i and $\pi_{\mathscr{C}} = \pi_1 \circ \cdots \circ \pi_m$. We also denote π for $\pi_{\mathscr{C}}$ when no confusion is likely. Observe that the isomorphism class of the surface $S_{\mathscr{C}}$ over S does not depend on the choice of the labelling of $\mathscr{C}$ with the previous property. Throughout this paper, we will consider constellations $\mathscr{C}$ such that $\pi_{\mathscr{C}}$ is a desingularization of (S,P). It follows from 1.1 that constellations of points infinitely near P and desingularizations of (S,P) are equivalent data. The constellation $\mathscr{C}_m$ such that $\pi_{\mathscr{C}_m}$ is the minimal desingularization of (S,P) is called the *minimal constellation for* (S,P).

In the above situation, we denote by $E^i_{i1}, \ldots, E^i_{is_i}$ the irreducible components of the exceptional locus of π_i (the upper i means that they are divisors on S_i), E^i_{ik} may not be a Cartier divisor on S_i but it is a Weil divisor. For $j > i$, let E^j_{ik} (resp. E^{*j}_{ik}) be the strict transform (resp. the total transform in the sense of Mumford [16]) of E^i_{ik} in S_j and, for simplicity, denote E_{ik} for E^m_{ik} and E^*_{ik} for E^{*m}_{ik}. Therefore, E_{ik} is an irreducible component of the exceptional locus of π and E^*_{ik} is a $\mathbb{Q}$-Cartier divisor on $S_{\mathscr{C}}$. We call $\Delta_{\mathscr{C}}$, or simply $\Delta = \{(i,k)/\, 1 \le i \le m,\ 1 \le k \le s_i\}$, the set of indices of the irreducible components of the exceptional locus of π.

Observe that the $\mathbb{Q}$-vector space $N^1(S_{\mathscr{C}}/S) = (\mathrm{Pic}(S_{\mathscr{C}})/\equiv) \otimes \mathbb{Q}$ (where $\mathrm{Pic}(S_{\mathscr{C}})$ denotes the Picard group of $S_{\mathscr{C}}$ and $\equiv$ is the numerical equivalence relation $D \equiv 0$ if $D.E_\gamma = 0$ for any exceptional curve E_γ in $S_{\mathscr{C}}$) is $\mathbf{E}_{\mathscr{C}} := \oplus_{\gamma \in \Delta} \mathbb{Q} E_\gamma = \oplus_{\gamma \in \Delta} \mathbb{Q} E^*_\gamma$. This follows immediately from the fact that the intersection matrix $(E_\alpha . E_\beta)_{\alpha,\beta \in \Delta}$ is negative definite.

We can consider total orders in the set of indices Δ compatible with the labelling in $\mathscr{C}$ in the sense that $(i,k) < (i',k')$ whenever $i < i'$, for any k, k'. These total orders will be called *enumerations of* Δ.

1.4. Definition. Let $\mathscr{C}$ be a constellation with origin at the point P of the rational surface singularity (S,P) and let ω be an enumeration of the set of indices Δ of the irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$. We define the *proximity matrix* of $\mathscr{C}$ with respect to ω to be the matrix $M_{\mathscr{C}\omega}$ of the change of basis from $\{E_\gamma^*\}_\gamma$ to $\{E_\gamma\}_\gamma$. That is,

$$M_{\mathscr{C}\omega}\underline{E}^* = \underline{E} \tag{2}$$

where by $\underline{E}$ and $\underline{E}^*$ we denote the column vectors consisting of the E_γ's and E_γ^*'s ordered by ω. We denote M for $M_{\mathscr{C}\omega}$ when no confusion is likely.

1.5. Remark. If (S,P) is nonsingular, the above matrix has been introduced by Du Val [7]. In this case, each point in $\mathscr{C}$ gives rise to a unique irreducible component E_i of the exceptional locus of $\pi_{\mathscr{C}}$, that is, the cardinal of Δ is equal to the number of points in $\mathscr{C}$. Fixed a labelling on the points in $\mathscr{C}$, say $\mathscr{C} = \{P_1,\dots,P_m\}$, we have $E_i = E_i^* - \sum p_{ij}E_j^*$ where $p_{ij} = 1$ if $i < j$ and $P_j \in E_i^j$, and $p_{ij} = 0$ otherwise. Following Enriques terminology, the relation $P_j \to P_i$ if $P_j \in E_i^j$ is called proximity relation. If we denote by Pr the upper triangular matrix $(p_{ij})_{i,j}$ then the proximity matrix is $M = \mathrm{Id} - \mathrm{Pr}$ and hence, it only depends on the proximity relations.

We now analyse the structure of the proximity matrix. To do so we introduce some definition and prove a preliminary result.

1.6. Definition. Let C be an effective Weil divisor on the rational surface singularity (S,P) and $\mathscr{C} = \{P_1,\dots,P_m\}$ a constellation with origin at P. Let E_γ be an irreducible component of the exceptional locus of $\pi_{\mathscr{C}}$ obtained by the blowing up of P_i, i.e. $\gamma = (i,k)$ for some k $(1 \le k \le s_i)$, and v_γ the valuation of the function field $K(S)$ induced by E_γ. Then, the strict transform $\overline{C}^{i-1}$ of C on the surface S_{i-1} is a $\mathbb{Q}$-Cartier divisor and hence, $e_\gamma(C) := v_\gamma(\overline{C}^{i-1})$ is a well-defined rational number. The rational numbers $\{e_\gamma = e_\gamma(C)\}_{\gamma\in\Delta}$ are called the *effective orders* (or orders) *of* C *in* $\mathscr{C}$.

Note that, in particular, if (S,P) is nonsingular, C is a curve on (S,P) and $\mathscr{C} = \{P_1,\dots,P_m\}$ a constellation with origin at P, then, for $1 \le i \le m$, $e_i = v_i(\overline{C}^{i-1})$ is the multiplicity of $\overline{C}^{i-1}$ at P_i.

1.7. Proposition. *Let C be an effective Weil divisor on (S,P) and $\mathscr{C}$ a constellation with origin at P. Let C^* and $\overline{C}$ be, respectively, the total and strict transform of C by $\pi_{\mathscr{C}}$ and $\{e_\gamma\}_\gamma$ the orders of C in $\mathscr{C}$. Then,*

$$C^* = \overline{C} + \sum e_\gamma E_\gamma^*. \tag{3}$$

Proof. First, suppose that C is a Cartier divisor on (S,P) and take $h \in \mathcal{O}_{S,P}$ defining C in (S,P). Then, the total transform C^{*1} of C in S_1 is the Cartier divisor on S_1 given by h and

$$C^{*1} = \overline{C}^1 + \sum_{k=1}^{s_1} v_{1k}(C)E_{1k}^{*1}. \tag{4}$$

If C is not a Cartier divisor, there exists $r \in \mathbb{N}$ such that rC is a Cartier divisor and hence, the above equality still holds. To complete the proof it is enough to apply induction on equality (4). □

1.8. Corollary. *Let $\mathscr{C}$ be a constellation with origin at P and fix an enumeration ω of Δ. For $1 \le i < j \le m$, let V_{ij} be the $(s_i \times s_j)$-matrix of rational numbers $V_{ij} = (e_{jt}(E_{ik}^i))_{k,t}$ and let V be the $(n \times n)$-upper triangular matrix consisting of $(V_{ij})_{i<j}$ and with zeroes elsewhere. Then, the proximity matrix is $M_{\mathscr{C}\omega} = \mathrm{Id} - V$.*

Moreover, for $i < j$, if $P_j \notin E_{ik}^{j-1}$ then $e_{jt}(E_{ik}^i) = 0$ for $1 \le t \le s_j$, i.e. the kth row of V_{ij} is zero. If $P_j \in E_{ik}^{j-1}$ then $e_{jt}(E_{ik}^i) \ne 0$ for $1 \le t \le s_j$, i.e. all the elements of the kth row of V_{ij} are nonzero.

Proof. First note that, from the proof of 1.7 it follows that equality (3) is also true for a Weil divisor C on a surface S with rational singularities, instead of a rational surface singularity (S,P). Now, to compute the matrix $M = M_{\mathscr{C}\omega}$ it suffices to apply (3) to each Weil divisor E_{ik}^i defined on S_i, that is, we suppose our surface S is S_i and consider the desingularization $S_{\mathscr{C}} \to S_i$. In this way, one has

$$E_{ik} = E_{ik}^* - \sum_{j>i} \sum_{1 \le t \le s_j} e_{jt}(E_{ik}^i)E_{jt}^*$$

and the first assertion is proved. The second part of the corollary follows from the fact that P_j is the center in S_{j-1} of the valuation v_{jt}, for $1 \le t \le s_j$. □

1.9. Definition. Given a constellation $\mathscr{C}$ with origin at P and two points P_i and P_j in $\mathscr{C}$, we say that P_j *is proximate to* P_i, and we denote it by $P_j \to P_i$ (or simply $j \to i$), if either P_j is in the first infinitesimal neighbourhood of P_i or else P_j lies on the strict transform of the first infinitesimal neighbourhood of P_i. That is, if a labelling in the sense of 1.3 is given, say $\mathscr{C} = \{P_1, \dots, P_m\}$, then P_j is proximate to P_i if and only if $j > i$ and $P_j \in E_{ik}^j$ for some $k, 1 \le k \le s_i$.

To each constellation $\mathscr{C}$ we associate a *tree* $\mathscr{T}_{\mathscr{C}}$, or simply $\mathscr{T}$, in the following way: the vertices of $\mathscr{T}$ are in a one to one correspondence with the points in $\mathscr{C}$, and the edges with the pairs (P_i, P_j) such that P_j is in the first infinitesimal neighbourhood of P_i. We can also associate to $\mathscr{C}$ a *tree with proximity relations* $\mathscr{T}_{\mathscr{C}}^P$, or $\mathscr{T}^P$. It consists of $\mathscr{T}$, together with some additional dotted lines corresponding to the pairs (P_i, P_j) whenever P_j is proximate to P_i but not in the first infinitesimal neighbourhood of P_i.

1.10. Remark. If (S,P) is nonsingular, knowing the tree with proximity relations is equivalent to knowing the proximity matrix. From Corollary 1.8 it follows that, in general, for a rational surface singularity the tree with proximity relations is obtained from the proximity matrix, but this matrix contains more information.

We now discuss the structure of the intersection matrix $\Lambda_{\mathscr{C}\omega} = (E^*_\alpha . E^*_\beta)_{\alpha,\beta}$.

1.11. We embed the rational surface singularity (S,P) in a germ of smooth variety (Y,P) by $\sigma : (S,P) \to (Y,P)$ (recall that the dimension of Y can be taken to be $r+1$, where r is the multiplicity of (S,P), see [1]). We consider the sequence of point blowing ups

$$\begin{array}{ccccccccc} Y_{\mathscr{C}} = Y_m & \xrightarrow{\pi_m} & Y_{m-1} & \xrightarrow{\pi_{m-1}} \cdots & \xrightarrow{\pi_2} & Y_1 & \xrightarrow{\pi_1} & Y_0 = Y \\ \sigma_m \uparrow & & \sigma_{m-1} \uparrow & & & \sigma_1 \uparrow & & \sigma \uparrow \\ S_{\mathscr{C}} = S_m & \xrightarrow{\pi_m} & S_{m-1} & \xrightarrow{\pi_{m-1}} \cdots & \xrightarrow{\pi_2} & S_1 & \xrightarrow{\pi_1} & S_0 = S \end{array}$$

where σ_i is the embedding of S_i in Y_i and $\boldsymbol{\pi}_{i+1}$ is the blowing up with center $\sigma_i(P_{i+1})$. We denote by $\mathbb{E}^i_i$ the exceptional divisor of $\boldsymbol{\pi}_i$ ($\mathbb{E}^i_i$ is isomorphic to $\mathbb{P}^r$ if $r+1$ is the dimension of Y). There exist strictly positive integers ρ_{ik} for $1 \le i \le m$ and $1 \le k \le s_i$ such that

$$\sigma_i^*(\mathbb{E}^i_i) = \rho_{i1} E^i_{i1} + \cdots + \rho_{is_i} E^i_{is_i}. \tag{5}$$

In fact, for each i, $Z_i = \sum_k \rho_{ik} E^*_{ik}$ is the fundamental cycle for the desingularization $S_{\mathscr{C}} \to S_{i-1}$ of (S_{i-1}, P_i) (see [1, Theorem 4]) and thus, the integers $\{\rho_\gamma\}_{\gamma \in \Delta_{\mathscr{C}}}$ do not depend on the embedding. The above relations give us some information about the matrix Λ.

1.12. Proposition. *Let $\mathscr{C}$ be a constellation with origin at P and fix an enumeration ω of $\Delta_{\mathscr{C}}$. If, for $1 \le i \le m$, Λ_i is the $(s_i \times s_i)$-matrix of rational numbers $\Lambda_i = (E^*_{ik}.E^*_{ir})_{k,r}$, then $\Lambda_{\mathscr{C}\omega}$ is the symmetric matrix consisting of the boxes Λ_i in the diagonal and zeroes elsewhere.*

Moreover, with the notation in (5), if $(\underline{\rho}_i)^t = (\rho_{i1}, \ldots, \rho_{is_i})$, then we have

$$(\underline{\rho}_i)^t \Lambda_i \underline{\rho}_i = -\mathrm{mult}_{P_i}(S_{i-1}). \tag{6}$$

In particular, if the point P_i is nonsingular then $s_i = 1$ and $\Lambda_i = -1$ and, if (S,P) is nonsingular, then $\Lambda = -\mathrm{Id}$.

Proof. The fundamental cycle $Z_i = \sum_k \rho_{ik} E^*_{ik}$ for $S_{\mathscr{C}} \to S_{i-1}$ is equal to $\sigma_m^*(\mathbb{E}^*_i)$, where $\mathbb{E}^*_i$ is the total transform of $\mathbb{E}^i_i$ by the morphism $Y_{\mathscr{C}} \to Y_i$. When $i \ne j$ we have $\mathbb{E}^*_i . \mathbb{E}^*_j = 0$ and hence, $0 = Z_i.Z_j = \sum_{k,t} \rho_{ik}\rho_{jt}(E^*_{ik}.E^*_{jt})$. Since $E^*_{ik}.E^*_{jt}$ is nonnegative for $i \ne j$ and all ρ_γ are strictly positive, whenever $i \ne j$ we have $E^*_{ik}.E^*_{jt} = 0$ and hence the first assertion is proved. Equality (6) follows from the fact that the multiplicity of the rational surface singularity (S_{i-1}, P_i) at P_i is $-Z_i.Z_i$ [1, Theorem 4]. □

1.13. Remark. Equality (5) insures that $\mathbb{E}_i^*.S_{\mathscr{C}} = \rho_{i1}E_{i1}^* + \cdots + \rho_{is_i}E_{is_i}^*$. However, the above assertion does not hold if we substitute the total transforms by the strict transforms. That is why the basis $\{E_\gamma^*\}_\gamma$ of $\mathbf{E}_{\mathscr{C}} \otimes \mathbb{Q}$ plays an important role.

For example, let (S,P) be the rational double point of type $\mathbf{D}_5$ defined by $x^4 + xy^2 + z^2 = 0$ in a neighbourhood of the point $P = (0,0,0)$ in $\mathbf{k}^3$ (where $\mathbf{k}$ is an algebraically closed field). Let $\mathscr{C}_m = \{P_1 = P, P_2, P_3, P_4\}$ be the constellation defining the minimal desingularization of (S,P), where P_2 and P_3 are points in the first infinitesimal neighbourhood of P_1 giving rise to the irreducible components E_{21}^2, E_{22}^2 and E_3^3 in the respective point blowing ups, and P_4 is the point of intersection $E_{21}^2 \cap E_{22}^2$, which defines only one irreducible component of the exceptional locus of $\pi_{\mathscr{C}_m}$. Let $\mathscr{C}$ be the constellation $\mathscr{C}_m \cup \{P_5\}$ where P_5 is a point in the first infinitesimal neighbourhood of P_1 such that $P_5 \notin \mathscr{C}_m$. If we consider the natural embedding of (S,P) in $Y = \mathbf{k}^3$, then the strict transform $\mathbb{E}_1$ of $\mathbb{E}_1^1$ in $Y_{\mathscr{C}}$ is given by $\mathbb{E}_1 = \mathbb{E}_1^* - \mathbb{E}_2^* - \mathbb{E}_3^* - \mathbb{E}_4^* - \mathbb{E}_5^*$ and we have $\mathbb{E}_1.S_{\mathscr{C}} = 2E_1^* - E_{21}^* - E_{22}^* - E_3^* - E_4^* - E_5^*$. However, the strict transform E_1 of E_1^1 by $\pi_{\mathscr{C}}$ is $E_1 = E_1^* - \frac{1}{2}(E_{21}^* + E_{22}^* + E_3^* + E_4^*) - E_5^*$, and hence

$$\mathbb{E}_1.S_{\mathscr{C}} = 2E_1 + E_5 \tag{7}$$

is different from $\rho_1 E_1 = 2E_1$.

In fact, in the same way as in 1.2 and 1.9, we may define points infinitely near or proximate to the point P viewed as points over the variety Y. In this way, the points infinitely near P over S are exactly the points infinitely near P over Y which lie on the corresponding strict transform of S. However, the notion of proximity is different if we consider the points over S or over Y. For example, equality (7) insures that every closed point P_6 in $E_5 - E_1$ is a point proximate to P viewed as points over the ambient space Y, but it is not proximate to P viewed as points over S. What we always have is that proximity over S implies proximity over the ambient space, since $E_{j1} \cup \cdots \cup E_{js_j} \subset \mathbb{E}_j.S_{\mathscr{C}}$.

Given a constellation $\mathscr{C}$ with origin at the point P of the rational surface singularity (S,P) and an enumeration ω of $\Delta_{\mathscr{C}}$, the intersection form on $\mathbf{E}_{\mathscr{C}}$ may be represented by two different matrices: $\Lambda_{\mathscr{C}\omega}$ in terms of the total transforms $\{E_\gamma^*\}$ and $(E_\alpha.E_\beta)_{\alpha,\beta}$ in terms of the strict transforms $\{E_\gamma\}$. Let us show the relationship between $\Lambda_{\mathscr{C}\omega}$ and $(E_\alpha.E_\beta)_{\alpha,\beta}$.

1.14. Theorem. *Let $\mathscr{C}$ be a constellation with origin at P and ω an enumeration of $\Delta_{\mathscr{C}}$; then we have $(E_\alpha.E_\beta)_{\alpha,\beta} = M_{\mathscr{C}\omega}\Lambda_{\mathscr{C}\omega}M_{\mathscr{C}\omega}^t$. Conversely, given any total order on the set of irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$, from the intersection matrix $(E_\alpha.E_\beta)_{\alpha,\beta}$ with respect to this order we can recover an enumeration ω and we can compute the proximity matrix $M_{\mathscr{C}\omega}$ and the intersection matrix $\Lambda_{\mathscr{C}\omega}$.*

Proof. The first equality is clear from the definitions. Now, given the matrix $(E_\alpha.E_\beta)_{\alpha,\beta}$, we can compute the fundamental cycle Z for the morphism $\pi_{\mathscr{C}}$, since Z is the minimal cycle with exceptional support such that $Z.E_\gamma \leq 0$ for each γ. Besides, Tjurina proved that $Z.E_{ik} = 0$ for $i \neq 1, 1 \leq k \leq s_i$ and $Z.E_{1k} \neq 0$ for $1 \leq k \leq s_1$ (see [19, Proposition 1.2]). Therefore, we can deduce which are the files of $(E_\alpha.E_\beta)_{\alpha,\beta}$ corresponding to the irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$ defined by the blowing up of P. Suppose these are the s_1 first files, then the $(n-s_1)\times(n-s_1)$-matrix obtained by erasing the s_1 first files and columns of $(E_\alpha.E_\beta)_{\alpha,\beta}$ defines a negative definite bilinear form whose dual graph has as many connected components as points in the first infinitesimal neighbourhood of P are in $\mathscr{C}$, that is, the space in which this bilinear form is defined is decomposed as orthogonal sum of subspaces. By reiterating the above process, we recover an enumeration ω of $\Delta_{\mathscr{C}}$ and we obtain the tree $\mathscr{T}_{\mathscr{C}}$ together with the assignation to each vertex v_i of $\mathscr{T}_{\mathscr{C}}$ the number s_i of irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$ defined by the blowing up of the point P_i corresponding to v_i.

Now, let us show how the proximity matrix $M = M_{\mathscr{C}\omega} = (m_{\alpha\beta})_{\alpha,\beta}$ is obtained from this information. We have $M = \mathrm{Id} - V$ where V is the upper triangular matrix consisting of $(V_{ij})_{i<j}$ (see 1.8). Let us compute the $\{V_{ij}\}_{1\leq i<j\leq m}$ in a recursive way. We know that

$$E^*_{i_0,r} = E_{i_0,r} - \sum_{i>i_0,1\leq s\leq s_i} m_{(i_0,r),(i,s)}E^*_{i,s},$$

where $\{m_{(i_0,r),(i,s)}\}$ is the unique solution of the system of equations defined by imposing $E^*_{i_0,r}.E^*_{i,s} = 0$ for $i > i_0$, $1 \leq s \leq s_i$. Thus, once we know $\{V_{ij}\}_{i>i_0}$, for $i > i_0$, we can write $E^*_{i,s}$ in terms of the E_γ's and hence, the solutions of the preceding system can be computed from the knowledge of $(E_\alpha.E_\beta)_{\alpha,\beta}$.

Finally, the intersection matrix $\Lambda = (E^*_\alpha.E^*_\beta)$ can be obviously described in terms of M and $(E_\alpha.E_\beta)_{\alpha,\beta}$. In fact, $\Lambda = M^{-1}(E_\alpha.E_\beta)_{\alpha,\beta}(M^t)^{-1}$. □

1.15. Remark. From the preceding result it follows that, given a constellation $\mathscr{C}$, the integers $\{\rho_\gamma\}_{\gamma\in\Delta_{\mathscr{C}}}$ can be computed from the proximity and intersection matrices M and Λ of $\mathscr{C}$ with respect to an enumeration ω, or equivalently, from the matrix $(E_\alpha.E_\beta)_{\alpha,\beta}$. In fact, to obtain $\{\rho_{1k}\}_{k=1}^{s_1}$ we just have to compute the fundamental cycle Z for $\pi_{\mathscr{C}}$ from the matrix $(E_\alpha.E_\beta)_{\alpha,\beta}$ and express it in terms of the E^*_γ's by the change of basis defined by M. For $i \geq 2$, the proximity and intersection matrices of the desingularization $S_{\mathscr{C}} \to S_{i-1}$ can be computed in a recursive way from M and Λ and hence, applying the above argument to the fundamental cycle Z_i for $S_{\mathscr{C}} \to S_{i-1}$, we obtain the integers $\{\rho_{ik}\}_{k=1}^{s_i}$.

2. Classification of curves in a rational surface singularity

In this section we extend the notion of equisingularity of germs of plane curves to germs of curves on a rational surface singularity (S,P). We prove as main result that if C and C' are equisingular curves in (S,P) then they are equiresoluble, that is, the respective multiplicities of the strict transforms of the branches of C and C' coincide.

2.1. Definition. Let C be a germ of reduced curve embedded in the rational surface singularity (S,P). An *embedded desingularization of* C *in* (S,P) is a desingularization $\pi : X \to S$ of (S,P) such that the strict transform $\overline{C}$ of C by π is nonsingular and the support of the total transform C^* has only normal crossings. By composing the minimal desingularization $\pi_{\mathscr{C}_m} : S_{\mathscr{C}_m} \to S$ of (S,P) with the minimal desingularization of the support of the total transform of C on $S_{\mathscr{C}_m}$, we get an embedded desingularization π of C in (S,P) which is minimal in the sense that it satisfies the universal property. The constellation $\mathscr{C}$ with origin at P such that $\pi = \pi_{\mathscr{C}}$ is called the *minimal constellation for* C *in* (S,P).

2.2. Definition. Let C be a germ of reduced curve in (S,P) and let $\mathscr{C}$ be the minimal constellation for C in (S,P). We define the *equisingularity class of* C *in* (S,P) to be the combinatorial object consisting of the weighted dual graph of $\pi_{\mathscr{C}}$ together with, for each $\gamma \in \Delta_{\mathscr{C}}$, an arrow with origin at the vertex corresponding to the divisor E_γ weighted by the number d_γ of analytic irreducible components of C whose strict transform intersects E_γ.

Given two germs C and C' of reduced curves embedded in (S,P), we say that C and C' are *equisingular in* (S,P) if their respective equisingularity classes in (S,P) coincide.

2.3. Definition. Let $\mathscr{C} = \{P_1,\dots,P_m\}$ be the minimal constellation for C in (S,P) and let $\{e_\gamma\}_{\gamma\in\Delta_{\mathscr{C}}}$ be the orders of C in $\mathscr{C}$. We define the *minimal weighted tree* $\mathscr{T}_S(C)$ *of* C *in* (S,P) to be the tree $\mathscr{T}_{\mathscr{C}}$ together with the weights $(e_{i1},\dots,e_{is_i})$ associated to the vertex of $\mathscr{T}_{\mathscr{C}}$ corresponding to P_i. Analogously, the *minimal weighted tree with proximity relations* $\mathscr{T}_S^P(C)$ *of* C *in* (S,P) consists of adding to $\mathscr{T}_S(C)$ the dotted lines corresponding to the proximity relations.

2.4. Proposition. *Let* C *and* C' *be equisingular germs of reduced curves in* (S,P). *Then, their respective minimal weighted trees with proximity relations coincide.*

Proof. Let $\mathscr{C}$ be the minimal constellation for C in (S,P). The total transform of C by $\pi_{\mathscr{C}}$ is $C^* = \overline{C} + \sum_\gamma e_\gamma E_\gamma^*$ (Proposition 1.7) and the number d_γ of analytic irreducible components of C whose strict transform intersects E_γ is $d_\gamma = \overline{C}.E_\gamma$. Thus, given an enumeration ω of $\Delta_{\mathscr{C}}$, we have $\underline{d} = -M\Lambda\underline{e}$ where M and Λ denote, respectively, the proximity matrix $M = M_{\mathscr{C}\omega}$ and the intersection matrix $\Lambda = \Lambda_{\mathscr{C}\omega}$ of $\mathscr{C}$ with respect to ω. Now, Theorem 1.14 insures that $((E_\alpha.E_\beta)_{\alpha,\beta},\underline{d})$ and $(M,\Lambda,\underline{e})$ are equivalent data. Therefore, two germs C and C' of reduced curves are equisingular in (S,P) if and only if, given the minimal constellations $\mathscr{C}$ and $\mathscr{C}'$ for C and C' in (S,P), there exist enumerations ω and ω' of $\Delta_{\mathscr{C}}$ and $\Delta_{\mathscr{C}'}$ such that $M_{\mathscr{C}\omega} = M_{\mathscr{C}'\omega'}$, $\Lambda_{\mathscr{C}\omega} = \Lambda_{\mathscr{C}'\omega'}$ and $\underline{e} = \underline{e}'$. In particular, it follows that if C and C' are equisingular in (S,P) then their respective minimal weighted trees with proximity relations $\mathcal{T}_S^P(C)$ and $\mathcal{T}_S^P(C')$ coincide. □

2.5. Remark. In the nonsingular case, the intersection matrix Λ is $-\mathrm{Id}$ and the proximity matrix defines and is defined by the proximity relations. Therefore, two germs C and C' of reduced curves embedded in a nonsingular germ of surface are equisingular if and only if their respective minimal weighted trees with proximity relations coincide. The minimal weighted tree with proximity relations of a curve C in a nonsingular surface has the same information as the sequences of multiplicities of the branches of C together with the intersection multiplicities of every two branches of C. Thus, Definition 2.2 coincides with the notion of equisingularity of curves if (S,P) is nonsingular (see [20]).

In general, the converse to Proposition 2.4 is not true as we show in the next example. Let (S,P) be a rational double point of type $\mathbf{A}_3$. The exceptional locus of the blowing up with center P has two irreducible components E_{11}^1 and E_{12}^1 and, if P_2 is the intersection point $E_{11}^1 \cap E_{12}^1$, then $\mathcal{C}_m = \{P_1 = P, P_2\}$ is the minimal constellation for (S,P). Let us take a point $P_3 \in E_{11}^1$, $P_3 \notin E_{12}^1$ and consider the constellation $\mathcal{C} = \mathcal{C}_m \cup \{P_3\}$.

Let $\{E_{11}, E_{12}, E_2, E_3\}$ be the enumerated irreducible components of the exceptional locus of $\pi_{\mathcal{C}}$ and let $\underline{d}^t = (0,1,2,2)$ and $(\underline{d}')^t = (0,0,4,1)$. For each $\gamma \in \Delta_{\mathcal{C}}$ we take d_γ (resp. d'_γ) distinct nonsingular irreducible algebroid curves in $S_{\mathcal{C}}$ transversal to E_γ and not intersecting E_α for $\alpha \neq \gamma$, and we define C (resp. C') to be the projection on (S,P) of this union of curves. Both C and C' are germs of reducible curves embedded in (S,P) whose minimal constellation is $\mathcal{C}$. Moreover, their respective equisingularity classes in (S,P) are as shown in Figs. 1 and 2 and hence, C and C' are not equisingular in (S,P).

However, their minimal weighted trees with proximity relations $\mathcal{T}_S^P(C)$ and $\mathcal{T}_S^P(C')$ coincide. In fact, if $\{e_\gamma\}_\gamma$ and $\{e'_\gamma\}_\gamma$ are respectively the orders of C and C' in $\mathcal{C}$, then we have $e_{11} = e'_{11} = 11/4, e_{12} = e'_{12} = 9/4, e_2 = e'_3 = 1$, $e_3 = e'_2 = 2$. Therefore, $\mathcal{T}_S^P(C) = \mathcal{T}_S^P(C')$ is as shown in Fig. 3.

2.6. Proposition. *Let C and C' be two equisingular germs of reduced curves in (S,P). If C is a Cartier divisor, then C' is a Cartier divisor.*

Proof. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the minimal constellation for C (resp. C') in (S,P) and $\{e_\gamma\}_\gamma$ (resp. $\{e'_\alpha\}_\alpha$) the orders of C in $\mathcal{C}$ (resp. of C' in $\mathcal{C}'$). Since C and C' are

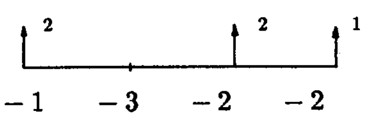

Fig. 1.

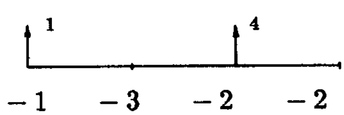

Fig. 2.

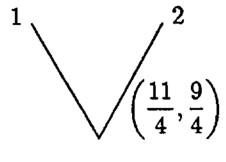

Fig. 3.

equisingular in (S,P), there exist enumerations ω and ω' of $\Delta_{\mathscr{C}}$ and $\Delta_{\mathscr{C}'}$ such that $M_{\mathscr{C}\omega} = M_{\mathscr{C}'\omega'}$, $\Lambda_{\mathscr{C}\omega} = \Lambda_{\mathscr{C}'\omega'}$ and $\underline{e} = \underline{e}'$. Let $\{E_\gamma\}_\gamma$ (resp. $\{E'_\gamma\}_\gamma$) be the irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$ (resp. $\pi_{\mathscr{C}'}$). Then, the respective total transforms of C and C' by $\pi_{\mathscr{C}}$ and $\pi_{\mathscr{C}'}$ are $C^* = \overline{C} + \sum b_\gamma E_\gamma$ and $C'^* = \overline{C}' + \sum b_\gamma E'_\gamma$ where $\underline{b} = (M^{\mathrm{t}}_{\mathscr{C},\omega})^{-1}\underline{e} = (M^{\mathrm{t}}_{\mathscr{C}',\omega'})^{-1}\underline{e}'$. If C is a Cartier divisor, then the b_γ's are integers and, applying to C'^* the result of Artin quoted in 1.1(b), we conclude that C' is a Cartier divisor. □

2.7. Proposition. *Let C and C' be two equisingular germs of reduced curves in (S,P). If $\mathscr{C}$ and $\mathscr{C}'$ are minimal constellations for C and C' in (S,P), then $\ell((\pi_{\mathscr{C}})_*\mathcal{O}_{\overline{C}}/\mathcal{O}_C) = \ell((\pi_{\mathscr{C}'})_*\mathcal{O}_{\overline{C}'}/\mathcal{O}_{C'})$, i.e. the δ-invariants $\delta(C)$ and $\delta(C')$ coincide.*

Proof. With the notation as in 2.6, let $D_C = \sum b_\gamma E_\gamma$ be the divisor with exceptional support for $\pi_{\mathscr{C}}$ defined by the total transform of C. Let $\lceil D_C \rceil$ be the minimal divisor with exceptional support for $\pi_{\mathscr{C}}$ such that $(D_C - \lceil D_C \rceil).E_\gamma \geq 0$ for all γ. Then we have $\delta(C) = \frac{1}{2}D_C.(-D_C + K_{S_{\mathscr{C}}}) + \frac{1}{2}e(C)$ where $K_{S_{\mathscr{C}}}$ is a canonical divisor on $S_{\mathscr{C}}$ and $e(C) = (D_C - \lceil D_C \rceil).(D_C - \lceil D_C \rceil - K_{S_{\mathscr{C}}})$ (see [9, Theorem 2.2]). Since (S,P) is a rational surface singularity, $p_a(E_\gamma) = 0$ ([2, Lemma 1.3]) and, aplying the adjunction formula to each E_γ we deduce that $E_\gamma.K_{S_{\mathscr{C}}} = -2 - (E_\gamma.E_\gamma)$. Therefore, $D_C.(-D_C + K_{S_{\mathscr{C}}})$ and $e(C)$ only depend on the equisingularity class of C in (S,P) and hence, $\delta(C) = \delta(C')$. □

Now, let us view the germs of reduced curves in (S,P) as germs of curves in an ambient nonsingular variety, and let us study its behaviour.

2.8. Definition. Let C be a germ of reduced curve centered at the point P. The normalization $n : \widetilde{C} \to C$ of C is the composition of a sequence of point blowing ups. Let $\mathscr{C}_0$ be the constellation of points infinitely near P over C so defined and $\mathscr{C}_0^*$ the constellation obtained by adding to $\mathscr{C}_0$ the closed points in $n^{-1}(P)$. In this way, the branches of the tree $\mathscr{T}_0(C)$ associated to $\mathscr{C}_0^*$ correspond bijectively to the branches of C at P. The *equiresolution class* of C is the combinatorial data $(\mathscr{T}_0(C), \underline{m})$ where $\underline{m}$ consists of a weight funtion $\underline{m}_B$ for each branch B of $\mathscr{T}_0(C)$. Each $\underline{m}_B$ is defined on the set of vertices of B and the $\underline{m}_B$-weight of the vertex of B corresponding to a point Q is the multiplicity at Q of the strict transform of the branch of C corresponding to B.

Given two germs of reduced curves C and C', we say that C and C' are *equiresoluble* if their respective equiresolution classes coincide.

Let us consider an embedding of the rational surface singularity (S,P) in a germ of smooth variety (Y,P). Let C be a germ of irreducible curve embedded in (S,P) and $\mathscr{C}$ the minimal constellation for C in (S,P). Then, following the notation in 1.11 and 1.12, we obtain the following result.

2.9. Lemma. *Given a point P_i in $\mathscr{C}$, if $(\underline{e}_i)^{\mathrm{t}} = (e_{i1},\dots,e_{is_i})$ are the orders of C at P_i, then we have*

$$\mathrm{mult}_{P_i}(\overline{C}^{i-1}) = \mathbb{E}_i^*.\overline{C} = Z_i.\overline{C} = -(\underline{\rho}_i)^{\mathrm{t}}\Lambda_i\underline{e}_i.$$

Proof. Given a curve embedded in a nonsingular variety, the multiplicity of this curve at a closed point P' is the intersection product of the strict transform of the curve with the exceptional divisor of the blowing up of P'. Therefore, the first equality holds. The other equalities follow from the facts that $\mathbb{E}_i^*.S_{\mathscr{C}} = Z_i$ and $E_{ik}^*.E_{i'k'}^* = 0$ whenever $i \neq i'$. □

2.10. Theorem. *Let C and C' be two equisingular germs of reduced curves in (S,P). Then, they are equiresoluble.*

Proof. Let $\mathscr{C}$ (resp. $\mathscr{C}'$) be the minimal constellation for C (resp. C') in (S,P) and $\{e_\gamma\}_\gamma$ (resp. $\{e'_\gamma\}_\gamma$) the orders of C in $\mathscr{C}$ (resp. of C' in $\mathscr{C}'$). There exist enumerations ω and ω' of $\Delta_{\mathscr{C}}$ and $\Delta_{\mathscr{C}'}$, respectively, such that $M_{\mathscr{C}\omega} = M_{\mathscr{C}'\omega'}$, $\Lambda_{\mathscr{C}\omega} = \Lambda_{\mathscr{C}'\omega'}$ and $\underline{e} = \underline{e}'$. Call these data $(M,\Lambda,\underline{e})$ and let $\underline{d} = -M\Lambda\underline{e}$. In this way, if $\{E_\gamma\}_\gamma$ and $\{E'_\gamma\}_\gamma$ are the irreducible components of the exceptional locus of $\pi_{\mathscr{C}}$ and $\pi_{\mathscr{C}'}$ respectively then, for each γ, d_γ is the number of analytic irreducible components of C (resp. C') whose strict transform intersects E_γ (resp. E'_γ).

From the matrices M and Λ we deduce the tree $\mathscr{T}$ associated to both $\mathscr{C}$ and $\mathscr{C}'$. In fact, the number of vertices of $\mathscr{T}$ is the number of nonzero boxes in Λ and the matrix M define the structure of tree (see 1.8 and 1.12). The d_γ's determine the number of branches of both C and C' and its relative situation in $\mathscr{T}$. Therefore, since the $\underline{m}$-weights of the equiresolution classes consist of a weight funtion for each branch, we may suppose, without loss of generality, that both C and C' have only one branch.

Now, we compute the integers $\{\rho_\gamma\}_\gamma$ from the matrices M and Λ as in 1.15, obtaining that the ρ_γ's are the same for both $\mathscr{C}$ and $\mathscr{C}'$. Therefore, Lemma 2.9 guarantees that, if P_i and P'_i are the points of $\mathscr{C}$ and $\mathscr{C}'$ corresponding to a vertex v_i in $\mathscr{T}$, then $\mathrm{mult}_{P_i}(\overline{C}^{i-1}) = \mathrm{mult}_{P'_i}(\overline{C}'^{i-1})$, that is, the $\underline{m}$-functions defined by C and C' in $\mathscr{T}$ coincide. Moreover, since C and C' are irreducible, there is a unique branch $\mathscr{T}'$ of $\mathscr{T}$ for which the $\underline{m}$-weights are nonzero. The tree $\mathscr{T}'_0$ obtained erasing the vertices of $\mathscr{T}'$ for which the multiplicity is 1 is the tree associated to the constellations $\mathscr{C}_0$ and $\mathscr{C}'_0$ defining the normalizations of C and C'. Thus, to compute the equiresolution class of both C and C', we consider $\mathscr{T}'_0$ together with the $\underline{m}$-function restricted to $\mathscr{T}'_0$, and we add a final vertex with multiplicity 1. Therefore, the equiresolution classes of C and C' coincide. □

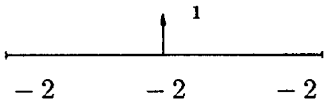

Fig. 4.

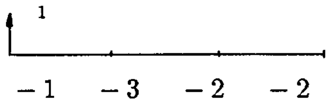

Fig. 5.

2.11. Remark. In the proof of the preceding theorem, we have computed explicitly the equiresolution class of a curve C embedded in (S,P) from the equisingularity class of C in (S,P). However, the converse to Theorem 2.10 is not true, as we show in next example.

Let (S,P) be a rational double point of type $\mathbf{A}_3$ and consider the minimal constellation $\mathscr{C}_m = \{P_1, P_2\}$ and the constellation $\mathscr{C} = \{P_1, P_2, P_3\}$ defined in 2.5. Let C (resp. C') be a germ of irreducible curve in (S,P) whose strict transform intersects transversally E_2 (resp. E_3). In the same way as in 2.5, C and C' are obtained by projecting a suitable algebroid curve in $S_{\mathscr{C}}$. The minimal constellations for C and C' in (S,P) are, respectively, $\mathscr{C}_m$ and $\mathscr{C}$ and their equisingularity classes in (S,P) are shown in Figs. 4 and 5.

Therefore, C and C' are not equisingular in (S,P). However, following the process described in the proof of 2.10, we deduce that both C and C' are nonsingular curves and hence, they are equiresoluble.

3. Minimal system of generators of a complete ideal

In this section we obtain a formula to calculate the minimal number of generators of an $\mathfrak{m}$-primary complete ideal I of the local ring $R = \mathcal{O}_{S,P}$ of a rational surface singularity (S,P), generalizing the formula given by Hoskin and Deligne when (S,P) is nonsingular. We also give an algorithm to describe a minimal system of generators of I, which is a generalization of the procedure of Casas [5].

3.1. Definition. A *cluster of points infinitely near P* (or cluster with origin at P) is a pair $K = (\mathscr{C}, \{v_\gamma\}_{\gamma \in \Delta_{\mathscr{C}}})$ where $\mathscr{C}$ is a constellation with origin at P and the v_γ's are nonnegative rational numbers in such a way that the sequence $(v_{i1}, \ldots, v_{is_i})$ is associated to the point P_i of $\mathscr{C}$. We call this sequence the *virtual orders of K at P_i* and we call the constellation $\mathscr{C}$ the *support* of K.

3.2. Remark. To each germ of reduced curve C in (S,P) we may associate a cluster $K(C) = (\mathscr{C}, \underline{e})$ where $\mathscr{C}$ is the minimal constellation for C in (S,P) and $\{e_\gamma\}_\gamma$ are the orders of C in $\mathscr{C}$. If C and C' are two germs of reduced curves in (S,P) such that $K(C) = K(C')$, then they are equisingular in (S,P). However, the converse is not true. For example, let C (resp. C') be a germ of irreducible curve in a double point of type $\mathbf{A}_3$ whose strict transform by the minimal desingularization of $\mathbf{A}_3$ intersects transversally E_{11} (resp. E_{12}), then C and C' are equisingular in (S,P) but $K(C)$ is different from $K(C')$.

3.3. Definition. Let $K = (\mathscr{C}, \underline{v})$ be a cluster with origin at P. We consider the $\mathbb{Q}$-Cartier divisor on $S_{\mathscr{C}}$ defined by $D_K := \sum_\gamma v_\gamma E_\gamma^*$. Given an effective Weil divisor C on (S,P), we say that C *goes through* K if and only if $C^* \geq D_K$ (i.e. $C^* - D_K$ is effective) where C^* is the total transform of C by $\pi_{\mathscr{C}}$. We say that C *goes through* K *with effective orders equal to the virtual ones* if and only if the orders of C in $\mathscr{C}$ are $\{v_\gamma\}_\gamma$, or equivalently, if $C^* = \overline{C} + D_K$ where $\overline{C}$ is the strict transform of C by $\pi_{\mathscr{C}}$.

The set of effective Weil divisors on (S,P) going through a given cluster K defines a family of cycles in (S,P). Analogously, we may consider the Cartier divisors going through K and define the following ideal of R:

$$I_K := \{0\} \cup \{h \in R - \{0\} \,/\, \text{the divisor } (h) \text{ goes through } K\}. \tag{8}$$

That is, I_K is the stalk at P of $(\pi_{\mathscr{C}})_*(\mathcal{O}_{S_{\mathscr{C}}}(-D_K))$, i.e. the set of all $h \in R$ such that the $\mathbb{Q}$-Cartier divisor $(h)^* - D_K$ is effective. Thus, it is a complete (i.e. integrally closed) and $\mathfrak{m}$-primary ideal, where $\mathfrak{m}$ is the maximal ideal of R.

Conversely, let us associate a cluster to each $\mathfrak{m}$-primary complete ideal.

3.4. Definition. Let I be an $\mathfrak{m}$-primary complete ideal of R. Let $\mathscr{C}$ be a constellation with origin at P such that the sheaf $I\mathcal{O}_{S_{\mathscr{C}}}$ is invertible, and D_I the divisor on $S_{\mathscr{C}}$ such that $I\mathcal{O}_{S_{\mathscr{C}}} = \mathcal{O}_{S_{\mathscr{C}}}(-D_I)$. The divisor D_I has exceptional support for the desingularization $\pi_{\mathscr{C}}$. Moreover, since I is finitely generated and the orders in $\mathscr{C}$ of any effective Cartier divisor (h) are nonnegative rational numbers, we have $D_I = \sum_\gamma v_\gamma E_\gamma^*$ for some nonnegative rational numbers $\{v_\gamma\}_\gamma$. We define the *cluster* K_I *with support in* $\mathscr{C}$ *associated to* I to be $K_I = (\mathscr{C}, \{v_\gamma\}_\gamma)$. By the completeness of I, we have $I = I_{K_I}$, i.e. I is the stalk at P of $(\pi_{\mathscr{C}})_*(\mathcal{O}_{S_{\mathscr{C}}}(-D_I))$ ([14, Proposition 6.2]).

Note that, if $\mathscr{C}'$ is another constellation such that $\pi_{\mathscr{C}'}$ factorices through $S_{\mathscr{C}}$, then $\Delta_{\mathscr{C}} \subset \Delta_{\mathscr{C}'}$ and the cluster with support in $\mathscr{C}'$ associated to I is $(\mathscr{C}', \{v'_\gamma\}_{\gamma \in \Delta_{\mathscr{C}'}})$ where $v'_\gamma = v_\gamma$ if $\gamma \in \Delta_{\mathscr{C}}$ and $v'_\gamma = 0$ otherwise.

In fact, if $K = (\mathscr{C}, \underline{v})$ is the cluster with support in $\mathscr{C}$ associated to I, then $\{v_\gamma\}_{\gamma \in \Delta_{\mathscr{C}}}$ are the orders in $\mathscr{C}$ of the Cartier divisor defined by a generic element of I and hence, there exist effective Cartier divisors going through K with effective orders equal to the virtual ones. Conversely, let us show that this condition characterizes the clusters K which are associated to some complete ideal. The following proposition is a generalization of the geometric theory of Enriques.

3.5. Proposition. *Given a cluster $K = (\mathscr{C}, \underline{v})$ with origin at P, the following conditions are equivalent*:

(i) *D_K is a divisor on $S_{\mathscr{C}}$ and $D_K.E_\gamma \leq 0$ for all $\gamma \in \Delta_{\mathscr{C}}$.*

(ii) *There exists an effective Cartier divisor C on (S,P) going through K with effective orders equal to the virtual ones.*

(iii) *There exist Cartier divisors as in* (ii), *and the only points infinitely near P through which all these Cartier divisors go are the points in $\mathscr{C}$.*

The clusters K satisfying the above conditions are called Cartier clusters.

Proof. (iii)$\Rightarrow$(ii) is obvious. To prove (ii)$\Rightarrow$(i), let C be a Cartier divisor satisfying (ii). Then $C^* = \overline{C} + D_K$ and hence, D_K is a divisor and $D_K.E_\gamma = -\overline{C}.E_\gamma \leq 0$ for all γ. Now, suppose K is a Cartier cluster, then $d_\gamma = -D_K.E_\gamma$ is a nonnegative integer. For every γ, we take d_γ nonsingular irreducible algebroid curves in $S_{\mathscr{C}}$ transversal to E_γ and not intersecting any of the E_α's, for $\alpha \neq \gamma$. We consider the union $\mathscr{E}$ of these curves and the divisor on $S_{\mathscr{C}}$ given by $D' = \mathscr{E} + D_K$. For any γ, $D'.E_\gamma = 0$ and hence, 1.1(b) guarantees the existence of an element $h \in R$ such that $(h)^* = D'$. The Cartier divisor C on (S,P) defined by h goes through K with effective orders equal to the virtual ones. If a different choice $\mathscr{E}'$ of the algebroid curves in $S_{\mathscr{C}}$ is made, then we obtain a new Cartier divisor C' on (S,P) going through K with effective orders equal to the virtual ones and such that the unique points infinitely near P on the strict transforms of C and C' are the points of $\mathscr{C}$. □

3.6. Corollary. *Given a constellation $\mathscr{C}$ with origin at P, there is a one to one correspondence between $\mathfrak{m}$-primary complete ideals I of R such that the sheaf $I\mathscr{O}_{S_{\mathscr{C}}}$ is invertible and Cartier clusters with support in $\mathscr{C}$.*

Proof. It follows from 3.5 because, given a Cartier cluster $K = (\mathscr{C}, \underline{v})$, D_K is a divisor on $S_{\mathscr{C}}$ and K is the cluster with support in $\mathscr{C}$ associated to the complete ideal defined by the stalk at P of $(\pi_{\mathscr{C}})_*(\mathscr{O}_{S_{\mathscr{C}}}(-D_K))$. □

Now, given an R-module M, we denote by $\ell(M)$ the length of M as R-module. Given an ideal I of R, the colength of I is denoted by $\ell(R/I)$. The following proposition will be used to compute the minimal number of generators of an $\mathfrak{m}$-primary complete ideal in terms of its associated cluster.

3.7. Proposition. *Let I be an $\mathfrak{m}$-primary complete ideal of R. Let $\mathscr{C}$ be a constellation with origin at P such that $I\mathscr{O}_{S_{\mathscr{C}}}$ is invertible, $K = (\mathscr{C}, \underline{v})$ the Cartier cluster with support in $\mathscr{C}$ associated to I and $D = D_K$ the corresponding divisor. If $K_{S_{\mathscr{C}}}$ is a canonical divisor on $S_{\mathscr{C}}$, and M and Λ are the proximity and intersection matrices of $\mathscr{C}$ with respect to some enumeration of $\Delta_{\mathscr{C}}$, then*

$$\ell(R/I) = -\tfrac{1}{2}D.(D + K_{S_{\mathscr{C}}}) = -\tfrac{1}{2}\underline{v}^{\mathrm{t}}\Lambda\underline{v} + \underline{v}^{\mathrm{t}}M^{-1}(1 + \tfrac{1}{2}(E_\gamma.E_\gamma))_\gamma. \tag{9}$$

Proof. Since (S,P) is a rational surface singularity, we have $\ell(R/I) = h^0(S_{\mathscr{C}}, \mathscr{O}_D) = \mathscr{X}(\mathscr{O}_D)$ (see [14, Lemma 23.1]). Therefore, the first equality in (9) follows from the adjunction formula. Besides, $D.D = \underline{v}^{\mathrm{t}}\Lambda\underline{v}$ and, for each γ, $E_\gamma.K_{S_{\mathscr{C}}} = -2-(E_\gamma.E_\gamma)$. Writing D in terms of the E_γ's by the change of basis given by M, we reach equality (9). □

3.8. Remark. When (S,P) is a germ of nonsingular surface, $\Lambda = -\mathrm{Id}$ and $E_i.E_i = -1 - \sum_{j \to i} 1$, that is, $2 + (E_i.E_i)$ is the sum of the elements of the ith row of M.

Therefore, if $K = (\mathscr{C}, \{v_i\}_i)$ is the cluster with support in $\mathscr{C}$ associated to I, then $\ell(R/I) = \frac{1}{2}\sum_i(v_i^2 + v_i)$. In fact, this is the formula given by Hoskin and Deligne in [10, Theorem 5.1; 6, Theorem 2.13].

3.9. Corollary. *Let (S,P) be a rational double point and $\mathscr{C}_m$ a minimal constellation for (S,P). Let I be an $\mathfrak{m}$-primary complete ideal of R such that $I\mathcal{O}_{S_{\mathscr{C}_m}}$ is invertible and let $K = (\mathscr{C}_m, \underline{v})$ be the cluster with support in $\mathscr{C}_m$ associated to I. Then,*

$$\ell(R/I) = -\tfrac{1}{2}\underline{v}^{t}\Lambda\underline{v}, \tag{10}$$

where Λ is an intersection matrix defined by $\mathscr{C}_m$.

Proof. In the above conditions, $E_\gamma.E_\gamma = -2$ for all γ (see [1]). Applying this to the right-side member of (9) we obtain equality (10). □

3.10. Theorem. *Let I be an $\mathfrak{m}$-primary complete ideal of R. Let $\mathscr{C}$ be a constellation with origin at P such that $I\mathcal{O}_{S_\mathscr{C}}$ is invertible, Z the fundamental cycle for the morphism $\pi_\mathscr{C}$ and $D = D_K$ the divisor on $S_\mathscr{C}$ associated to I. Then, the minimal number of generators $\mu(I)$ of I is given by*

$$\mu(I) = \ell(I/\mathfrak{m}I) = -D.Z + 1. \tag{11}$$

Moreover, if $K = (\mathscr{C}, \underline{v})$ is the Cartier cluster with support in $\mathscr{C}$ associated to I and $Z = \sum_{k=1}^{s_1} \rho_{1k}E_{1k}^*$, then

$$\mu(I) = 1 - \sum_{1\le k,r\le s_1} v_{1k}\rho_{1r}(E_{1k}^*.E_{1r}^*). \tag{12}$$

Therefore, $\mu(I)$ only depends on the orders $(v_{11},\dots,v_{1s_1})$ of the origin of K. In fact, $\mu(I) = 1 + \mathrm{mult}_P(C)$ where C is the Cartier divisor defined by a generic element of I.

Proof. By Nakayama's lemma, given $x_1,\dots,x_r$ elements of I, they are a minimal system of generators of I if and only if its classes $\{\bar{x}_1,\dots,\bar{x}_r\}$ in $I/\mathfrak{m}I$ are a basis of the **k**-vector space $I/\mathfrak{m}I$ and hence, $\mu(I) = \dim_{\mathbf{k}}(I/\mathfrak{m}I) = \ell(I/\mathfrak{m}I)$. Since $D + Z$ is the divisor on $S_\mathscr{C}$ associated to the complete ideal $\mathfrak{m}I$, applying 3.7 we obtain

$$\mu(I) = \ell(I/\mathfrak{m}I) = \ell(R/\mathfrak{m}I) - \ell(R/I) = -D.Z - \tfrac{1}{2}Z.(Z + K_{S_\mathscr{C}}),$$

where $K_{S_\mathscr{C}}$ is a canonical divisor on $S_\mathscr{C}$. Since $p_a(Z) = 0$, we have $Z.(Z + K_{S_\mathscr{C}}) = -2$ and (11) is proved. Equality (12) follows from (11) since $D = \sum v_\gamma E_\gamma^*$, and the last assertion is consequence of Lemma 2.9. □

3.11. Remark. If (S,P) is a germ of nonsingular surface and I is an $\mathfrak{m}$-primary complete ideal of $\mathcal{O}_{S,P}$, then $Z = E_1^*$ and $\mu(I) = v_1 + 1$, where v_1 is the multiplicity of I at P (see [11, Theorem 2.1]).

3.12. Corollary. *Let (S,P) be a rational double point. Let $K = (\mathscr{C}, \underline{v})$ be a Cartier cluster with origin at P and I_K the complete ideal defined by K. Then, the minimal number of generators of I_K is given by*

$$\mu(I_K) = \begin{cases} v_{11} + v_{12} + 1 & \text{if } (S,P) \text{ is of type } \mathbf{A}_n \ (n \geq 2), \\ 2v_1 + 1 & \text{if } (S,P) \text{ is of type } \mathbf{A}_1, \\ v_1 + 1 & \text{if } (S,P) \text{ is of type } \mathbf{D}_n \ (n \geq 4), \mathbf{E}_6, \mathbf{E}_7 \text{ or } \mathbf{E}_8. \end{cases}$$

Proof. If (S,P) is an $\mathbf{A}_n$-singularity ($n \geq 2$), then the exceptional locus of the blowing up with center P has two irreducible components E_{11}^1 and E_{12}^1. Besides, for any constellation $\mathscr{C}$, the fundamental cycle for the desingularization $\pi_{\mathscr{C}}$ is $Z = E_{11}^* + E_{12}^*$ and we have $E_{11}^*.E_{11}^* = E_{12}^*.E_{12}^* = -n/(n-1)$ and $E_{11}^*.E_{12}^* = 1/(n-1)$. Therefore, the orders of K at the origin are a pair (v_{11}, v_{12}) and $\mu(I_K) = v_{11} + v_{12} + 1$ in this case. Analogously, a Cartier cluster K on an $\mathbf{A}_1$-singularity has only one order v_1 at the origin, and the fundamental cycle for $\pi_{\mathscr{C}}$ is $Z = E_1^*$ where $E_1^*.E_1^* = -2$, therefore $\mu(I_K) = 2v_1 + 1$. Finally, if (S,P) is a singularity of type $\mathbf{D}_n$ ($n \geq 4$), $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$, then the exceptional locus of the blowing up with center P has only one irreducible component and, for any constellation $\mathscr{C}$, the fundamental cycle for $\pi_{\mathscr{C}}$ is $Z = 2E_1^*$ where $E_1^*.E_1^* = -\frac{1}{2}$. Therefore, a Cartier cluster K on (S,P) has one order v_1 at the origin, and $\mu(I_K) = v_1 + 1$. □

Now, let K be any cluster with origin at P. We will give an algorithm to describe a minimal system of generators of the complete ideal I_K. First, let us prove a preliminary result which is a generalization of Laufer's procedure to compute the fundamental cycle [12, Proposition 4.1].

3.13. Lemma. *Let $\mathscr{C}$ be a constellation with origin at P and $D = \sum_\gamma b_\gamma E_\gamma$ a $\mathbb{Q}$-Cartier divisor on $S_{\mathscr{C}}$. Then, among all the exceptional divisors D' on $S_{\mathscr{C}}$ such that $D' \geq D$ and $D'.E_\gamma \leq 0$ for all γ, there is a minimal one $\overline{D}$.*

We can compute $\overline{D}$ in the following recurrent way. Let $D_1 = \sum_\gamma b_\gamma^1 E_\gamma$ where b_γ^1 is the smallest integer such that $b_\gamma^1 \geq b_\gamma$. Having defined D_t, if $D_t.E_\gamma \leq 0$ for all γ, then $\overline{D} = D_t$. Otherwise, we take γ_t such that $D_t.E_{\gamma_t} > 0$ and we set $D_{t+1} = D_t + E_{\gamma_t}$.

Proof. Let $\mathscr{D}$ be the set of divisors D' on $S_{\mathscr{C}}$ with exceptional support such that $D' \geq D$ and $D'.E_\gamma \leq 0$ for all γ. If Z is the fundamental cycle for $\pi_{\mathscr{C}}$ then $rZ \in \mathscr{D}$ for $r \gg 0$ and hence, $\mathscr{D}$ is nonempty. Besides, let $D_1' = \sum_\gamma c_\gamma^1 E_\gamma$ and $D_2' = \sum_\gamma c_\gamma^2 E_\gamma$ belong to $\mathscr{D}$, and let $D' = \sum_\gamma c_\gamma E_\gamma$ where $c_\gamma = \inf\{c_\gamma^1, c_\gamma^2\}$. Fixed γ, let us suppose $c_\gamma^1 \leq c_\gamma^2$, then $D'.E_\gamma = c_\gamma^1(E_\gamma.E_\gamma) + \sum_{\alpha \neq \gamma} c_\alpha(E_\alpha.E_\gamma) \leq D_1'.E_\gamma \leq 0$ and, since $D' \geq D$, it follows that D' belongs to $\mathscr{D}$. Therefore, there exists a minimal element $\overline{D}$. To prove the second assertion, it is enough to show that the divisors D_t defined recurrently from D satisfy $D_t \leq \overline{D}$. It is clear that $D_1 \leq \overline{D}$. Suppose that $D_t < \overline{D}$, then $D_{t+1} = D_t + E_{\gamma_t}$ where $(\overline{D} - D_t).E_{\gamma_t} < 0$. Thus, the effective divisor $\overline{D} - D_t$ contains E_{γ_t}, i.e. $D_{t+1} \leq \overline{D}$, and the lemma is proved. □

3.14. Let $K = (\mathscr{C}, \underline{v})$ be a cluster with origin at P. Let us apply the preceding procedure to the $\mathbb{Q}$-Cartier divisor D_K. The divisor $\overline{D}$ so obtained satisfies $\overline{D}.E_\gamma \le 0$ for all γ and hence, the argument in the proof of 3.5 insures that, in the expression $\overline{D} = \sum_\gamma v_\gamma^0 E_\gamma^*$ of $\overline{D}$ in terms of the E_γ^*'s, the rational numbers $\{v_\gamma^0\}_\gamma$ are nonnegative. Thus, it follows from the definition of $\overline{D}$ that the cluster $K_0 = (\mathscr{C}, \underline{v}^0)$ is the unique Cartier cluster with support in $\mathscr{C}$ such that $I_{K_0} = I_K$. We call it the *Cartier cluster induced by K*.

Let K' be the cluster associated to the complete ideal $\mathfrak{m}I_K$, that is, K' is the cluster with support in $\mathscr{C}$ whose virtual orders at the points of $\mathscr{C}$ different from P are the virtual orders $\{v_\gamma^0\}$ of K_0 and whose orders at P are $\{v_{1k}^0 + \rho_{1k}\}_{k=1}^{s_1}$. Let us construct a sequence of Cartier clusters $\{K_t\}_{t=0}^r$, where r is the minimal number of generators of I_K, such that $K_r = K'$. We define inductively K_{t+1} from K_t in the following way: "Let $K_t = (\mathscr{C}_t, \underline{v}^t)$ and $K' = (\mathscr{C}_t, \underline{v}')$ where $\mathscr{C}_t$ is a constellation which is a common support of K_t and K'. We fix an enumeration of the set of indices $\Delta_{\mathscr{C}_t}$ and we choose E_α exceptional on $S_{\mathscr{C}_t}$ such that $v_\alpha^t < v_\alpha'$ and $v_\beta^t = v_\beta'$ for all $\beta < \alpha$ (since $I_{K'} \subset I_{K_t}$ there exists such an E_α). We take a point $Q_t \in E_\alpha$ such that $Q_t \notin E_\beta$ for $\beta \ne \alpha$ and consider the cluster $K_{t+1}' = (\mathscr{C}_t \cup \{Q_t\}, \{v_\gamma^t\}_{\gamma \in \Delta_{\mathscr{C}_t}} \cup \{1\})$. Then, K_{t+1} is the Cartier cluster induced by K_{t+1}', which is obtained applying Lemma 3.13."

3.15. Theorem. *The preceding algorithm constructs a sequence $\{K_t\}_{t=0}^r$ of Cartier clusters such that $I_{K_0} = I_K$, K_r is the cluster associated to the complete ideal $\mathfrak{m}I_K$ and, for each t, $I_{K_{t+1}} \subset I_{K_t}$ and $\ell(I_{K_t}/I_{K_{t+1}}) = 1$. If, for each t, we take $h_t \in I_{K_t} \setminus I_{K_{t+1}}$, then $\{h_t\}_{t=0}^{r-1}$ is a minimal system of generators of the ideal I_K.*

Proof. It is clear that the algorithm finishes after a finite number of steps. Besides, by the definition of K_{t+1} we have $I_{K_{t+1}} = I_{K_{t+1}'} \subset I_{K_t}$. Moreover, $I_{K_t}/I_{K_{t+1}}$ is isomorphic to $\mathcal{O}_{S_{\mathscr{C}_t}, Q_t}/\mathfrak{m}_{S_{\mathscr{C}_t}, Q_t}$ and hence, $\ell(I_{K_t}/I_{K_{t+1}}) = \dim(\mathcal{O}_{S_{\mathscr{C}_t}, Q_t}/\mathfrak{m}_{S_{\mathscr{C}_t}, Q_t}) = 1$. Thus, $r = \ell(I_K/\mathfrak{m}I_K)$ is the minimal number of generators of I_K and the theorem is proved. □

3.16. Remark. Since $\ell(I_{K_t}/I_{K_{t+1}}) = 1$ and both K_t and K_{t+1} are Cartier clusters, we can take the element h_t in $I_{K_t} \setminus I_{K_{t+1}}$ in such a way that the Cartier divisor C_{h_t} defined by h_t goes through K_t with effective orders equal to the virtual ones and such that the constellation $\mathscr{C}_t$ support of K_t gives rise to an embedded desingularization of C_{h_t} in (S, P). Therefore, the algorithm to compute the sequence $\{K_t\}_{t=0}^r$ determines the equisingularity classes of the curves $\{C_{h_t}\}_t$.

3.17. Example. Let (S, P) be a rational double point of type $\mathbf{A}_3$ defined by $xy + z^4 = 0$ in a neighbourhood of the point $P = (0,0,0)$ in $\mathbf{k}^3$. Let $\mathscr{C}_m = \{P_1, P_2\}$ be the minimal constellation for (S, P) (see 2.5) and K the cluster with support in $\mathscr{C}_m$ whose orders are $(1, \frac{1}{2})$ at P_1 and $\frac{5}{4}$ at P_2. We have $D_K = E_{11} + \frac{1}{2}E_{12} + 2E_2$ and hence, K is not a Cartier cluster. We apply Lemma 3.13 to D_K and we obtain the Cartier cluster $K_0 = (\mathscr{C}_m, \{(1,1), 1\})$ induced by K. Thus, Corollary 3.12 insures that the minimal number of generators of the complete ideal $I_K = I_{K_0}$ is $r = 3$.

The Cartier cluster $K' = K_3$ is obtained from K_0 by adding the orders $(1,1)$ at the origin, i.e. $K_3 = (\mathscr{C}_m, \{(2,2),1\})$. To construct K_1, we observe that $v_{11}^0 = 1 < 2 = v'_{11}$. We consider a closed point $Q_0 \in E_{11}$ such that $Q_0 \notin E_{12}, E_2$, and the cluster K'_1 with support in $\mathscr{C}_m \cup \{Q_0\}$ whose weights are $\{(1,1),1,1\}$. After applying Lemma 3.13, we obtain the Cartier cluster K_1 induced by K'_1. In fact, $K_1 = (\mathscr{C}_m, \{(2,1),\frac{1}{2}\})$ and the element h_0 of R defined by y belongs to $I_{K_0} \setminus I_{K_1}$. Analogously, to compute K_2 we observe that $v_{12}^1 = 1 < 2 = v'_{12}$ and, applying the same method, we obtain the Cartier cluster $K_2 = (\mathscr{C}_m, \{(2,2),0\})$. The class h_1 of x in R belongs to $I_{K_1} \setminus I_{K_2}$ and the element h_2 defined by z^2 belongs to $I_{K_2} \setminus I_{K_3}$. Therefore, $\{y, x, z^2\}$ define a minimal system of generators of the complete ideal I_K.

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